Chapter 10 - Lecture 2
The independent two sample t-test and confidence interval

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Review

Let

- $X_1, \ldots, X_m \sim N(\mu_1, \sigma_1^2)$
- $Y_1, \ldots, Y_n \sim N(\mu_2, \sigma_2^2)$
- The two samples are independent.

Last lecture: we have seen how to handle the case of known population variances and the case of unknown population variances when both $n > 40, m > 40$.

Today: we will see what happens when we have unknown population variances with at least one of $n \leq 40, m \leq 40$.

Two cases

There are two different cases:

- Special case: Pooled case (assuming $\sigma_1^2 = \sigma_2^2$).
- General case: Unpooled case (no assumptions on variance)

1. Unpooled case
   - Confidence Intervals
   - Hypothesis Testing

2. Pooled case
   - Introduction
   - Confidence Intervals
   - Hypothesis Test

3. Summary

Distribution

Statistics:

$$T = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{m} + \frac{S_2^2}{n}}}$$

Task: to find the distribution of the above random variable.

- In previous lecture we have seen that if both $n > 40, m > 40$ this follows $N(0, 1)$.
- If we have $n \leq 40$ or $m \leq 40$ then this follows t_ν
- How do we calculate the degrees of freedom ν?

Degrees of freedom ν

- The degrees of freedom are found if we round **down** to the nearest integer the following:

$$\nu = \frac{\left(\frac{s_1^2}{m} + \frac{s_2^2}{n}\right)^2}{\frac{\left(\frac{s_1^2}{m}\right)^2}{m-1} + \frac{\left(\frac{s_2^2}{n}\right)^2}{n-1}}$$

- The nickname of this formula is "the smile face".

$(1-\alpha)100\%$ Confidence Intervals

- A $(1-\alpha)100\%$ CI for $\mu_1 - \mu_2$ is

$$\bar{x} - \bar{y} \pm t_{\nu, \frac{\alpha}{2}} \sqrt{\frac{s_1^2}{m} + \frac{s_2^2}{n}}$$

- Remember, degrees of freedom ν is found by using the "smile face" formula.

Example10.6 (J. Agr. Food Chem., 2010: 8768C6775)

Which way of dispensing champagne,

1. the traditional vertical method
2. tilted beer-like pour (slanted pour)

would preserve more of the tiny gas bubbles that improve flavor and aroma? The following data was reported in the article On the Losses of Dissolved CO2 during Champagne Serving.

Temperature (°C)	Type of Pour	*n*	Mean (g/L)	SD
18	Traditional	4	4.0	.5
18	Slanted	4	3.7	.3
12	Traditional	4	3.3	.2
12	Slanted	4	2.0	.3

http://web.1.c2.audiovideoweb.com/1c2web3536/champagnepouring.pdf

Example10.6 (cont.)

Assuming that the sampled distributions are normal, calculate confidence intervals for the difference between true average dissolved CO2 loss for the traditional pour and that for the slanted pour at each of the two temperatures.

- For the 18 C temperature,

the number of degrees of freedom for the interval is:

$$\mathrm{df} = \frac{(\frac{0.5^2}{4} + \frac{0.3^2}{4})^2}{\frac{1}{3}(\frac{0.5^2}{4})^2 + \frac{1}{3}(\frac{0.3^2}{4})^2}$$

Rounding down, the CI will be based on 4 df. For a confidence level of 99%, use $t_{0.005,4} = 4.604$.

Example10.6 (cont.)

The 99% CI is:

$$\begin{aligned} & (4.0-3.7) \pm (4.604)\sqrt{\frac{0.5^2}{4}+\frac{0.3^2}{4}} \\ = & \ 0.3 \pm 4.604 \times 0.2915 = (-1.0, 1.6) \end{aligned}$$

Thus we can be highly confident that $-1.0 < \mu1 - \mu2 < 1.6$, where $\mu1$ and $\mu2$ are true average losses for the traditional and slant methods, respectively. Notice that this CI contains 0, so at the 99% confidence level, it is plausible that $\mu1 - \mu2 = 0$.

Hypothesis test

Null Hypothesis: $H_0 : \mu_1 - \mu_2 = \Delta_0$
Test statistic :

$$T = \frac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\frac{S_1^2}{m} + \frac{S_2^2}{n}}} \sim t_\nu$$

Rejection Regions:

- If $H_A : \mu_1 - \mu_2 > \Delta_0$, $t \geq t_{\nu,\alpha}$
- If $H_A : \mu_1 - \mu_2 < \Delta_0$, $t \leq -t_{\nu,\alpha}$
- If $H_A : \mu_1 - \mu_2 \neq \Delta_0$, $t \leq -t_{\nu,\alpha/2}$ and $t \geq t_{\nu,\alpha/2}$

Degrees of freedom ν is found by "smile face" formula

Example 10.7 (J. Mater. Civil Eng., 1996: 94C100)

The deterioration of many municipal pipeline networks across the country is a growing concern. One technology proposed for pipeline rehabilitation uses a flexible liner threaded through existing pipe. The article "Effect of Welding on a High-Density Polyethylene Line" reported the following data on tensile strength of liner specimens both when a certain fusion process was used and when this process was not used. The authors of the article stated that the fusion process increased the average tensile strength.

No fusion	2748	2700	2655	2822	2511			
	3149	3257	3213	3220	2753			
	$m = 10$	$\bar{x} = 2902.8$	$s_1 = 277.3$					
Fused	3027	3356	3359	3297	3125	2910	2889	2902
	$n = 8$	$\bar{y} = 3108.1$	$s_2 = 205.9$					

http://ascelibrary.org/doi/pdf/10.1061/(ASCE)0899-1561(1996)8

Example 10.7 (cont.)

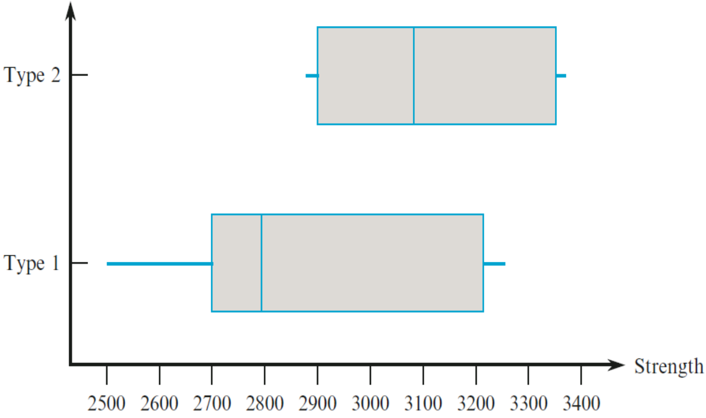

Figure 10.4 A comparative boxplot of the tensile strength data

Example 10.7 (cont.)

1. $H_0 : \mu_1 - \mu_2 = 0$ versus $H_1 : \mu_1 - \mu_2 < 0$, where μ_1 is the true average tensile strength of specimens when the no-fusion treatment is used and μ_2 denotes the true average tensile strength when the fusion treatment is used.
2. The null value is $\delta_0 = 0$, the test statistic is $T = \dfrac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\frac{S_1^2}{m} + \frac{S_2^2}{n}}}$
3. Under the H_0, $T \sim t_v$, where v is calculated by the "smile face" formula $\rightarrow v = 15.94$. Rounding down, take $v = 15$. hence $T \sim t_{15}$.
4. Plug in the value into the test statistics T will get the value of test statistic $t = -1.8$. The p-value is $P(T < t | T \sim t_{15}) = P(T < -1.8 | T \sim t_{15}) = 0.046$
5. If using $\alpha = 0.05$, then by $0.046 < 0.05$, reject the null hypothesis in favor of the alternative hypothesis, confirming the conclusion stated in the article at the level of 0.05.

In many examples, although the variances of the two populations are unknown, they can be assumed to be equal.

- For example, the number of credits male students and female students have each semester, might have different mean but it feels it is legitimate to assume that the two populations have equal variance.

Under the assumption that $\sigma_1^2 = \sigma_2^2$, we can get easy and exact results; Otherwise, we only have "nasty" and approximate results.

In this case, we have

$$T = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{S_p^2}{m} + \frac{S_p^2}{n}}} = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{S_p^2 \left(\frac{1}{m} + \frac{1}{n}\right)}}$$

Think:

- How do we calculate S_p^2?
- What is the distribution of T above?

Pooled variance and distribution

- S_p^2 is calculated as follows:

$$S_p^2 = \frac{(m-1)S_1^2 + (n-1)S_2^2}{n+m-2}$$

- Distribution of T:

$$T = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{S_p^2\left(\frac{1}{m} + \frac{1}{n}\right)}} \sim t_{m+n-2}$$

Why .....

Why .....

When we assume $\sigma_1^2 = \sigma_2^2$.

$$\bar{X} - \bar{Y} \sim N(\mu_1 - \mu_2, \sigma^2(\frac{1}{m} + \frac{1}{n}))$$

$$\frac{(m-1)S_1^2}{\sigma_1^2} \sim \chi^2_{m-1}, \quad \frac{(n-1)S_2^2}{\sigma_2^2} \sim \chi^2_{n-1}$$

$$\sigma_1^2 = \sigma_2^2 \rightarrow \frac{(m-1)S_1^2 + (n-1)S_2^2}{\sigma_2^2} \sim \chi^2_{m+n-2}$$

.

And we know $(\bar{X}, \bar{Y})$ is independent of (S_1^2, S_2^2).

Therefore,

$$T = \frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{S_p^2(\frac{1}{m} + \frac{1}{n})}} \sim t_{m+n-2}$$

where

$$S_p^2 = \frac{m-1}{m+n-2} S_1^2 + \frac{n-1}{m+n-2} S_2^2$$

is the **pooled estimator** of σ^2.

$(1-\alpha)100\%$ Confidence Intervals

A $100(1-\alpha)\%$ pooled two-sample t confidence interval for $\mu_1 - \mu_2$ (under equal variance assumption) is

$$(\bar{x}-\bar{y}-t_{\alpha/2,m+n-2}\sqrt{s_p^2(\frac{1}{m}+\frac{1}{n})}, \bar{x}-\bar{y}+t_{\alpha/2,m+n-2}\sqrt{s_p^2(\frac{1}{m}+\frac{1}{n})}).$$

Pooled two-sample t test

Null Hypothesis: $H_0 : \mu_1 - \mu_2 = \Delta_0$
Test statistic value:

$$t = \frac{\bar{X} - \bar{Y} - (\Delta_0)}{\sqrt{S_p^2 \left(\frac{1}{m} + \frac{1}{n} \right)}} \sim t_{m+n-2}$$

Rejection Regions:

- If $H_A : \mu_1 - \mu_2 > \Delta_0$, $t \geq t_{m+n-2,\alpha}$
- If $H_A : \mu_1 - \mu_2 < \Delta_0$, $t \leq -t_{m+n-2,\alpha}$
- If $H_A : \mu_1 - \mu_2 \neq \Delta_0$, $t \leq -t_{m+n-2,\alpha/2}$ and $t \geq t_{m+n-2,\alpha/2}$

Summary

Tag: Two independent Normal distributions with unknown variance, small sample size

Under assumption $\sigma_1^2 = \sigma_2^2$, $\quad T = \dfrac{\bar{X} - \bar{Y} - (\Delta_0)}{\sqrt{S_p^2\left(\dfrac{1}{m} + \dfrac{1}{n}\right)}} \sim t_{m+n-2}$

Summary

Tag: Two independent Normal distributions with unknown variance, small sample size

Under assumption $\sigma_1^2 = \sigma_2^2$, $\quad T = \dfrac{\bar{X} - \bar{Y} - (\Delta_0)}{\sqrt{S_p^2 \left(\dfrac{1}{m} + \dfrac{1}{n}\right)}} \sim t_{m+n-2}$

Without assumption $\sigma_1^2 = \sigma_2^2$, $\quad T = \dfrac{\bar{X} - \bar{Y} - \Delta_0}{\sqrt{\dfrac{S_1^2}{m} + \dfrac{S_2^2}{n}}} \sim t_\nu$ Degrees of freedom ν is found by "smile face" formula