Chapter 6 - Lecture 4
Distributions based on a normal random sample

Yuan Huang

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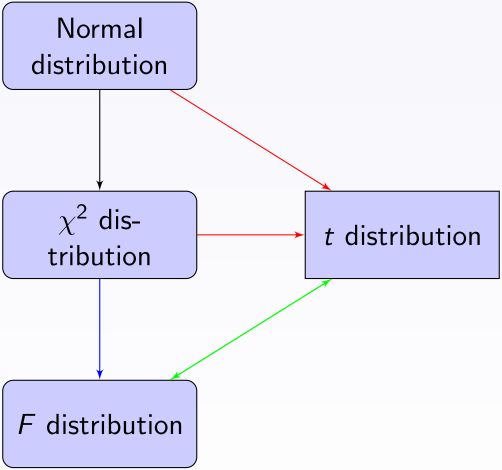
Normal distribution
χ^2 distribution
t distribution
F distribution

Definition of Chi-square distribution

- Chi-square distribution is fully determined by one parameter called degree of freedom ν, denoted as χ^2_ν.
- Chi-square distribution is a special case of Gamma distribution.

$$\chi^2_\nu = \text{Gamma}(\nu/2, 2)$$

- If $X \sim \chi^2_\nu$, then the pdf of X is

$$f(x) = \begin{cases} \frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{(\nu/2)-1} e^{-x/2} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

What is the connection with normal random variable ?

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Proposition

If $Z \sim N(0,1)$, *then* $X = Z^2 \sim \chi_1^2$

Try!

- If given $X \sim N(\mu, \sigma^2)$, can you define a random variable that follows chi-square distribution?
- If given random sample $X_1, \ldots, X_n$, can we define a random variable that follows chi-square distribution using $\bar{X}$?

Proof:

Properties of Chi-square distribution

1. If $X \sim \chi^2_\nu$, then $E(X) = \nu$;
2. If $X \sim \chi^2_\nu$, then $V(X) = 2\nu$;
3. If $X_1 \sim \chi^2_{\nu_1}$ and $X_2 \sim \chi^2_{\nu_2}$ and $X_1 \perp\!\!\!\perp X_2$ then $X_1 + X_2 \sim \chi^2_{\nu_1+\nu_2}$
4. If $X_3 = X_1 + X_2$, with $X_1 \sim \chi^2_{\nu_1}$, $X_3 \sim \chi^2_{\nu_3}$, $\nu_3 > \nu_1$ and $X_1 \perp\!\!\!\perp X_2$ then $X_2 \sim \chi^2_{\nu_3-\nu_1}$

Corollary

If $Z_1, \ldots, Z_n$ are i.i.d and $Z_1 \sim N(0,1)$, then $X = \sum_{i=1}^{n} Z_i^2 \sim \chi_n^2$.

Remark. This is an alternative definition of χ_ν^2 when ν is an integer.

Sample variance

- In previous lectures we have defined the sample mean $\bar{X}$. If we have a random sample $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$ then we have that $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$
- Now, we define the sample variance

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2$$

Theorem: Let a random sample $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$, then

1. $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$
2. $\bar{X}$ and S^2 are independent
3. $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

Scratch proof

Note that if $X_1, \ldots, X_n$ are i.i.d with $X_1 \sim N(\mu, \sigma^2)$, then we have $\frac{X_1-\mu}{\sigma}, \ldots, \frac{X_n-\mu}{\sigma}$ are i.i.d standard normal rv's.
Also, $\frac{(n-1)S^2}{\sigma^2} = \sum_{i=1}^{n} \frac{(X_i-\bar{X})^2}{\sigma^2}$ is really close to $\sum_{i=1}^{n}(\frac{X_i-\mu}{\sigma})^2$
To prove that $\dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$ we need the following two results:

- If $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$ is a random sample then $\bar{X} \perp\!\!\!\perp S^2$.
- If $X_3 = X_1 + X_2$ with $X_1 \sim \chi^2_{v_1}$, $X_3 \sim \chi^2_{v_3}$, $v_3 > v_1$ and $X_1 \perp\!\!\!\perp X_2$ then $X_2 \sim \chi^2_{v_3-v_1}$

Definition of t - distribution

- If $Z \sim N(0,1)$, $X \sim \chi^2_v$ and $X \perp\!\!\!\perp Z$ then

$$T = \frac{Z}{\sqrt{\frac{X}{v}}} \sim t_v$$

- The above is called t-**distribution with** v **degrees of freedom**.
- It is also known as the **"Student's t distribution"**

Question

- If $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$ a random sample, can you find the distribution of

$$W = \frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}}$$

Proposition

If $X_1, \ldots, X_n$ are i.i.d with $X_1 \sim N(\mu, \sigma^2)$, then

$$\frac{\sqrt{n}(\bar{X} - \mu)}{S} \sim t_{n-1}.$$

Definition of F distribution

- If $X_1 \sim \chi^2_{v_1}$, $X_2 \sim \chi^2_{v_2}$ and $X_1 \perp\!\!\!\perp X_2$ then:

$$F = \frac{\frac{X_1}{v_1}}{\frac{X_2}{v_2}} \sim F_{v_1, v_2}$$

- The above is called F **distribution with** v_1 **and** v_2 **degrees of freedom**.

Question

1. If a random variable $U \sim t_v$ can you find the distribution of $V = U^2$?
2. Suppose we have a random sample of size m from normal distribution $N(\mu_1, \sigma_1^2)$, and an independent random sample of size n form normal distribution $N(\mu_2, \sigma_2^2)$. Denote S_1^2 and S_2^2 as the sample variance from each group. What is the distribution of $\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2}$?

Proposition

If $X_1, \ldots, X_n$ are i.i.d with $X_1 \sim N(\mu, \sigma^2)$, then

$$\frac{n(\bar{X}_n - \mu)^2}{S^2} \sim F_{1,n-1}.$$

Summary:

When $X_1, \ldots, X_n$ are i.i.d normally distributed with mean μ and variance σ^2, we have the following results:

1. $\bar{X} \sim N(\mu, \sigma^2/n)$;
2. $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$;
3. $\bar{X}$ and S^2 are independent;
4. $\frac{\sqrt{n}(\bar{X}-\mu)}{S} \sim t_{n-1}$;
5. $\frac{n(\bar{X}-\mu)^2}{S^2} \sim F_{1,n-1}$.

Summary:
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5. $\frac{n(\bar{X}-\mu)^2}{S^2} \sim F_{1,n-1}$.

We will come back to other properties of these distributions time by time.

Homework for this session

Part of HW 2:

- Section 6.4 page 320 48, 50