Chapter 7 - Lecture 2 (2)
Maximum Likelihood Estimator

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1. For Bernoulli population, if you have two options: $p = 0.2$ and $p = 0.8$. The data you collect : $0, 1, 0, 0, 1, 0, 0, 0, 1, 0$. Which value of p will you choose?

2. For Normal population $N(\mu, 0.5^2)$, if you have two options : $\mu = -5$ and $\mu = 5$. The data you collect: $-4.5, -5.5, -5.1, -3.9, -6.1, -6.5, -5.3, -4.9, -4.7, -5.1$. Which value of μ will you choose?

Definition

Maximum Likelihood Estimation: To select the parameter that makes the event mostly likely to occur.

Think: How to measure the "likely" ?

Likelihood Function

If the pdf (pmf) function in the population is $f(X,\theta)$, and $X_1,\ldots,X_n$ is a random sample from the population. Then the likelihood function $L(\theta)$ is

$$L(\theta) = f(X_1,\theta)\cdot f(X_2,\theta)\cdot\ldots\cdot f(X_n,\theta) = \prod_{i=1}^{n} f(X_i,\theta).$$

Comments: Likelihood function represents how likely an event (a sample) will occur under distribution $f(X,\theta)$.

Maximum Likelihood Estimator:

Definition

Maximum Likelihood Estimator: $\hat{\theta} = \arg\max L(\theta)$, that is, $L(\hat{\theta}) = \max L(\theta)$.

Invariance Principle of MLE's

If we have $\hat{\theta}_1, \ldots, \hat{\theta}_m$ are MLE for parameters $\theta_1, \ldots, \theta_m$.
If $h(\theta_1, \ldots, \theta_m)$ is any function of $\theta_1, \ldots, \theta_m$.

$\rightarrow h(\hat{\theta}_1, \ldots, \hat{\theta}_m)$ is MLE for $h(\theta_1, \ldots, \theta_m)$

Large sample behavior of the MLE's

Under mild assumptions on the joint distribution of the sample, When the sample size is large,

1. $\hat{\theta}_{\mathrm{MLE}}$ is close to θ (consistent)
2. $\hat{\theta}_{\mathrm{MLE}}$ is approximately unbiased ($E(\hat{\theta}_{\mathrm{MLE}}) \approx \theta$)
3. $\hat{\theta}_{\mathrm{MLE}}$ has variance that is nearly as small as can be achieved by any unbiased estimator.

Step by step procedure on how to find MLE estimators

- **Step 1**: Find the likelihood function $L(\theta; x) = \prod_{i=1}^{n} f(X_i, \theta)$.
- **Step 2**: Find the natural logarithm of the likelihood function $l(\theta) = l(\theta; x) = \log L(\theta; x)$.
- **Step 3**: Take a derivative of $l(\theta)$ for each of the parameter. (If you have m parameters you need m derivatives).
- **Step 4**: Equalize each of the derivative with 0.
- **Step 5**: Solve the equations to find solutions. The solutions are the MLE estimators for the parameters.

[**Example 7.17**] Let $X_1, X_2, \ldots, X_n$ be a random sample from exponential distribution with parameter λ such that $f(x) = \lambda e^{-\lambda x}$. Find the MLE for λ.

[**Example 7.18**] Let $X_1, X_2, \ldots, X_n$ be a random sample from normal distribution $N(\mu, \sigma^2)$. Find the MLE for μ and σ^2.

Example 7.23

Suppose the waiting time for a bus is uniformly distributed on $[0, \theta]$ and the results $x_1, \ldots, x_n$ has the density $f(x; \theta) = \frac{1}{\theta}$ for $0 \leq x \leq \theta$ and 0 otherwise.

Homework 4

Uploaded under " Assignments " TAG.