Chapter 7 - Lecture 2 (1)
Method of Moments

Yuan Huang

Jan 30, 2013

Population moments and Sample moments

Let $X_1, \ldots, X_n$ be a random sample from any distribution $f(x)$.

- The k^{th} **population moment** :

$$\mathrm{E}(X^k)$$

- The k^{th} **sample moment** :

$$\frac{1}{n}\sum_{i=1}^{n} X_i^k$$

Moment estimator

Let $X_1, \ldots, X_n$ be a random sample from any distribution $f(x)$ which has m unknown parameters $\theta_1, \ldots, \theta_m$.

Definition

The **moment estimators** $\hat{\theta}_1, \ldots, \hat{\theta}_m$ are obtained by equating the first m sample moments to the corresponding m population moments and then solve for $\theta_1, \ldots, \theta_m$.

1. Identify how many parameters we need to estimate.(Let's say m).
2. Find the first m population moments:

$$E(X), E(X^2)\ldots, E(X^m)$$

3. Find the first m sample moments:

$$\frac{1}{n}\sum_{i=1}^{n} X_i, \quad \frac{1}{n}\sum_{i=1}^{n} X_i^2, \ldots, \quad \frac{1}{n}\sum_{i=1}^{n} X_i^m$$

4. Equalize each of the population moments to the corresponding sample moment.

$$\begin{aligned} E(X) &= \frac{1}{n}\sum_{i=1}^{n} X_i \\ E(X^2) &= \frac{1}{n}\sum_{i=1}^{n} X_i^2 \\ &\ldots \\ E(X^m) &= \frac{1}{n}\sum_{i=1}^{n} X_i^m \end{aligned}$$

5. The solutions for the above equations are the moment estimators for the parameters.

Example 1

Let $X_1, \ldots, X_n$ be random sample from Exponential(λ) such that

$$f(x) = \lambda e^{-\lambda x} \text{ for } x \geq 0$$

1. Given $E(X) = 1/\lambda$, find the moment estimator for λ.
2. If $x_1 = 3, x_2 = 7, x_3 = 5$, what is the estimate for λ?

Example 1 (Sol)

1. 1. The only parameter is $\lambda \rightarrow$ we need to find one equation.
   2. The population first moment is given as $E(X) = 1/\lambda$.
   3. The sample first moment is $\frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$.
   4. Let population 1^{st} moment $=$ sample 1^{st} moment
   $\rightarrow E(X) = \bar{X}$
   5. Solve and get $1/\hat{\lambda} = \bar{X} \rightarrow \hat{\lambda} = \frac{1}{\bar{X}_n}$
2. Plug in the observations $\rightarrow$ estimate $\hat{\lambda} = \frac{1}{5}$

Example 2

Let $X_1, \dots, X_n$ be random sample from Gamma(α, β) such that

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad \forall x \geq 0$$

1. Given $E(X) = \alpha\beta$ and $E(X^2) = \beta^2(\alpha+1)\alpha$, find the moment estimator for α, β.
2. If we have observations $152, 115, 109, 94, 88, 137, 152, 77, 160, 165, 125, 40, 128, 123, 136, 101, 62, 153, 83, 69,$ what are the estimates for α, β?

Example 2 (Sol)

1.
   1. We have two parameters $\alpha, \beta \rightarrow$ need two equations.
   2. The fist two population moments are given as
   $$E(X) = \alpha\beta \text{ and } E(X^2) = \beta^2(\alpha+1)\alpha.$$
   3. The fist two sample moments are
   $$\frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X} \text{ and } \frac{1}{n}\sum_{i=1}^{n} X_i^2$$
   4. Let population moments = corresponding sample moments
   $$\bar{X} = \alpha\beta, \qquad \frac{1}{n}\sum_{i=1}^{n} X_i^2 = \beta^2(\alpha+1)\alpha$$
   5. Solve and get
   $$\hat{\alpha} = \frac{\bar{X}^2}{\frac{1}{n}\sum X_i^2 - \bar{X}^2}, \qquad \hat{\beta} = \frac{\frac{1}{n}\sum X_i^2 - \bar{X}^2}{\bar{X}}$$

Example 2 (Sol)

2. If we have observations 152, 115, 109, 94, 88, 137, 152, 77, 160, 165, 125, 40, 128, 123, 136, 101, 62, 153, 83, 69, then calculate $\bar{X} = 113.5$ and $\frac{1}{n}\sum_{i=1}^{n} X_i^2 = 14087.8$

   Plug the values into $\hat{\alpha} = \frac{\bar{X}^2}{\frac{1}{n}\sum X_i^2 - \bar{X}^2}$, $\quad \hat{\beta} = \frac{\frac{1}{n}\sum X_i^2 - \bar{X}^2}{\bar{X}}$.

   - The estimate for α is $\hat{\alpha} = \frac{113.5^2}{14087.8 - 113.5^2} = 10.7$
   - The estimate for β is $\hat{\beta} = \frac{14087.8 - 113.5^2}{113.5} = 10.6$

R code for calculation (see 7.14 under Tag computation)

Example 3

Let $X_1, \ldots, X_n$ be random sample from Generalized negative binomial (r, p).

1. Given $E(X) = r(1-p)/p$ and $E(X^2) = r(1-p)/p^2$, find the moment estimator for r, p.
2. If we have the following table of observations

Goals	0	1	2	3	4	5	6	7	8	9	10
Frequency	29	71	82	89	65	45	24	7	4	1	3

What are the estimates for r, p?

Example 3 (Sol)

1.
1. We have two parameters $r, p \to$ need two equations.
2. The fist two population moments are given as
$$E(X) = r(1-p)/p \text{ and } E(X^2) = r(1-p)/p^2 \text{ .}$$
3. The fist two sample moments are
$$\frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X} \text{ and } \frac{1}{n}\sum_{i=1}^{n} X_i^2$$
4. Let population moments = corresponding sample moments
$$\bar{X} = r(1-p)/p, \qquad \frac{1}{n}\sum_{i=1}^{n} X_i^2 = r(1-p)/p^2$$
5. Solve and get
$$\hat{p} = \frac{\bar{X}}{\frac{1}{n}\sum X_i^2 - \bar{X}^2}, \qquad \hat{r} = \frac{\bar{X}^2}{\frac{1}{n}\sum X_i^2 - \bar{X}^2 - \bar{X}}$$

Example 3 (Sol)

2. If we have observations

Goals	0	1	2	3	4	5	6	7	8	9	10
Frequency	29	71	82	89	65	45	24	7	4	1	3

Note that this is a frequency table, with total sample size $= 29 + 71 + 82 + 89 + 65 + 45 + 24 + 7 + 4 + 1 + 3 = 420.$
Calculate $\bar{X} = 2.98$ and $\frac{1}{n}\sum_{i=1}^{n} X_i^2 = 12.4$
Plug the values into
$\bar{X} = r(1-p)/p, \qquad \frac{1}{n}\sum_{i=1}^{n} X_i^2 = r(1-p)/p^2$.

- The estimate for p is $\hat{p} = \frac{2.98}{12.40 - 2.98^2} = 0.85$
- The estimate for r is $\hat{r} = \frac{2.98^2}{12.40 - 2.98^2 - 2.98} = 16.5$

R code for calculation (see 7.15 under Tag computation)